Adaptive Euler-Heun method

1. What is the purpose to adaptive approximations of IVPs?

Answer: To ensure that we get an answer that is sufficiently accurate without unnecessary additional steps.

2. Why does an adaptive algorithm describe the error in terms of an acceptable error per unit time?

Answer: The error is given per unit time, as we may not know *apriori* how far we want to approximate the solution to the IVP. Consequently, it may be easier to say that we are willing to accept an error of 10^{-5} per second of the simulation, and then if we approximate the solution at $y(20)$ when the initial time is $t_0 = 0$, then the error will be no greater than 20×10^{-5} or 2×10^{-4}.

3. Summarize adaptive algorithms.

Answer:

a. Using the current value of h, approximate the solution at $t_k + h$ using two different algorithms, one that is generally less accurate and another that is generally more accurate. Denote these approximations by y and z, respectively.
b. Overestimate the error of the approximate y using $2|y - z|$.
c. The error for a single step of the less accurate method will be of the form Ch^m for an appropriate power of m. Thus, we have that $Ch^m \approx 2|y - z|$, so $C \approx 2|y - z|/h^m$.
d. Now, if we are willing to accept an error of ε_{abs} per unit time, then we are willing to accept $h\varepsilon_{\text{abs}}$ for that interval of width h.
e. Now, if the error of y is greater than $h\varepsilon_{\text{abs}}$, the we should try again with a smaller value of h; however, if the error of y is less than $h\varepsilon_{\text{abs}}$, our step size is too small and we could have used a larger value of h. In the first case, we'll try again. In the second case, we'd like to use a larger value of h.
f. What we really want is a scalar multiple of the interval width ah that gives the maximum acceptable error. Thus, we want to find an a so that $C(ah)^m = (ah)\varepsilon_{\text{abs}}$. We can substitute in the value of C above to get that $2|y - z|/h^m \times (ah)^m = (ah)\varepsilon_{\text{abs}}$. We can solve this for a as follows:

$$a = \sqrt[m-1]{\frac{h\varepsilon_{\text{abs}}}{2|y-z|}}.$$

g. If $a \leq 1$, we should try again with $0.9ah$, but if $a > 1$, we will use z to approximate the solution at $t_k + h$ and with the next step we will use a step size of $0.9ah$.

3. In our simpler example, we used Euler and Heun. Why not use Euler and the 4$^{\text{th}}$-order Runge-Kutta method, instead?

Answer: This would require four additional function evaluations per function evaluation for Euler. Consequently, a significant amount of extra work is required. The Dormand-Prince method specifically uses two orthogonal approximations that overlapped in a number of the required function evaluations, thereby reducing the work necessary.

4. Given the IVP, use $h = 0.01$, use the Euler-Heun adaptive method and allow a maximum error per unit time of 0.01. What would be an appropriate step size?

$$y^{(1)}(t) = -3ty(t)$$
$$y(0) = 2$$

Answer: $y = 2$, $z = 1.9997$, so $2|y - z| = 0.0006$ and thus we may calculate that $a = 0.16667$, so we really should repeat this with $0.9ah = 0.0015$, but this reduces h by a factor of more than 2, so we should instead try again with $h = 0.005$.

5. What happens if we use this new step size $h = 0.0015$ anyways?

Answer: $y = 2$, $z = 1.99999325$, so $2|y - z| = 0.0000135$ and thus we may calculate that $a = 1.111111111$.

6. Why is $a = 1.111111111$ and not 1.0?

Answer: We used $0.9ah$, just to be safe and use a step size less than ah means that with the next calculation, it is likely that $a > 1$, thus ensuring the error is sufficiently small. Note that $1.111111111 = 0.9^{-1}$.

7. In the course, we required that we did not reduce h by more than a factor of two. Why was this?

Answer: In general, it's a poor idea to make too significant a change in the step size. It may have been a fluke that a is either really large or really small, so it is safer to not change the step size h by more than a factor of two in either way.

Acknowledgement: Nikola Kubatlija who noted Questions 4 and 5 could use clarification, as it is only in Question 6 that we indicate that h should not be stretched or shrunk by more than a factor of two, and then noted a second error in my correction.